University *of Ljubljana*
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Solitary matter-waves in Bose-Einstein condensates

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Abstract

This seminar discusses solitary-matter waves in Bose-Einstein condensates, a nonlinear phenomenon that exhibits soliton-like properties. We start with theoretical description of condensate behaviour by mean-field Gross-Pitaevskii equation, explore its collapse instabilities and show exact soliton solution in 1D limit. In experiments, the residual three-dimensionality and external potentials alter fundamental soliton properties and introduce interesting new effects. The second part of the seminar focuses on experimental formation of single and multiple stable solitary waves and solitary wave collisions. In conclusion some examples of possible applications are given.

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1 Introduction

Solitons are non-dispersive localized wavepackets, well-known especially for appearing in shallow water and optics. They appear as a solution to equations in several one-dimensional (1D) systems, and are characterized by maintaining shape and amplitude while propagating and interacting with other solitons. This is achieved when **nonlinearity** of the medium cancels the effects of **dispersion**. The term soliton is usually reserved for solutions of partial differential equations describing physical systems which are exactly solvable (i.e. have integrable equations).

The first recorded observation of soliton was in 1834 in canal with shallow water. It was described as fast, well-defined and travelling without change of shape or speed. In the following years experiments in a wave tank have been made, demonstrating solitons passing through one another unchanged. In 1895, Korteweg–de Vries equation was derived, describing waves on shallow water with exact soliton solution that could describe said experiments. Nowadays solitons are best known in nonlinear optics, especially temporal solitons, whose existence was proposed in 1973. Soliton research has been conducted in diverse fields with solitons being suggested to describe proteins, DNA, plasma waves and so on. [1]

In this seminar similar phenomenon - **solitary matter-waves** - is presented, occurring in Bose-Einstein condensates (BECs) of ultracold atomic gases. BEC is a state of matter in which a macroscopic number of atoms share the same quantum wavefunction, implying that they behave coherently as a single matter-wave. Experiments can only approach the 1D limit needed for realization of true solitons, but solitary waves as their 3D analogue maintain a lot of key properties, such as propagation without dispersion on macroscopic distances. The nonlinearity which counteracts dispersion comes from interatomic interactions, which can be repulsive or attractive, the latter leading to solitary waves.

Solitary waves manifest in condensates as localised density peaks and were experimentally first realised in 2002 [2]. There is still ongoing research on this topic, since experimental observations of solitary waves triggered a lot of theoretical interest which in turn motivated a handful of experiments. A multitude of interesting questions have appeared and since cold atom systems can be very precisely manipulated, many ideas presented in theoretical proposals can be experimentally realised. In this seminar some elementary tools for theoretical description of such systems will be presented, followed by an interpretation of solitary waves formation and collision experiments.

2 Theoretical description

Bose-Einstein condensate is a state of matter of low density atomic gas, made of bosons, cooled to temperatures close to absolute zero. As we know, there is no limit to how many bosons can occupy a certain quantum state, so under appropriate conditions a large fraction of atoms goes into ground energy state of the system. Particles in a BEC all occupy the same quantum state, therefore it is assumed that they can be described by a single wavefunction, and so they behave like a single coherent matter-wave.

2.1 The Gross-Pitaevskii equation

The wavefunction of BEC can generally be described by the Gross-Pitaevskii equation (GPE), which has the form of nonlinear Schrödinger equation (NLS). It is assumed that only the **ground state** is occupied and that the gas has **low density** and weak interparticle interactions, which can be described by a **mean-field** approximation. [3]

Standard many-body Hamiltonian for interacting bosons in external potential $V_{ext}(\mathbf{r})$, written in second quantisation with boson field operators $\Psi(\mathbf{r})$ and $\Psi^\dagger(\mathbf{r})$, is

$$H = \int \mathrm{d}^3\mathbf{r} \left[\frac{\hbar^2}{2m} \boldsymbol{\nabla}\Psi^\dagger(\mathbf{r})\boldsymbol{\nabla}\Psi(\mathbf{r}) + V_{ext}(\mathbf{r})\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}) + \frac{1}{2}\int \mathrm{d}^3\mathbf{r}'\, \Psi^\dagger(\mathbf{r})\Psi^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\Psi(\mathbf{r}')\Psi(\mathbf{r}) \right],$$

where m is the atomic mass. Using the contact interatomic interaction

$$V(\mathbf{r}-\mathbf{r}') = g\,\delta(\mathbf{r}-\mathbf{r}')$$

and mean-field approximation we arrive to the total energy functional

$$E\left[\psi(\mathbf{r})\right] = \int \mathrm{d}^3\mathbf{r} \left[\frac{\hbar^2}{2m}|\boldsymbol{\nabla}\psi(\mathbf{r})|^2 + V_{ext}(\mathbf{r})\,|\psi(\mathbf{r})|^2 + \frac{1}{2}g\,|\psi(\mathbf{r})|^4 \right].$$

Here, $\psi(\mathbf{r})$ for the macroscopic wavefunction of the condensate, $|\psi(\mathbf{r})|^2$ is atomic density and the normalization is

$$\int |\psi(\mathbf{r})|^2 \mathrm{d}^3\mathbf{r} = N,$$

where N is the number of atoms in BEC. We then minimize the energy with respect to variations in $\psi(\mathbf{r})$ to get the time-independent GPE or minimize the action for the time-dependent version of the **Gross-Pitaevskii equation**:

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V_{ext}(\mathbf{r})\,\psi(\mathbf{r},t) + g\,|\psi(\mathbf{r},t)|^2\psi(\mathbf{r},t).$$

The coupling constant g is known from scattering theory as $g = \frac{4\pi\hbar^2}{m}a_s$, where a_s is the scattering length. This parameter describes the strength of atomic interaction, which is **repulsive** for $a_s > 0$ and **attractive** for $a_s < 0$. Since BEC has a very low density, scattering length is much smaller than the interparticle distances.

2.2 Stability of the condensate

It is instructive to consider the conditions for stability and collapse of BECs. The simplest way to see this is by taking the above equations for number of atoms and energy, generalising them for D dimensions:

$$N = \int |\psi(\mathbf{r})|^2 \mathrm{d}^D\mathbf{r}, \qquad E = \int \mathrm{d}^D\mathbf{r} \left[\frac{\hbar^2}{2m}|\boldsymbol{\nabla}\psi(\mathbf{r})|^2 + V_{ext}(\mathbf{r})\,|\psi(\mathbf{r})|^2 + \frac{1}{2}g_D\,|\psi(\mathbf{r})|^4 \right]$$

and perform a dimensional analysis. [4] We assume that L is the typical size of BEC and estimate the wavefunction from the first expression as $|\psi| \sim (\frac{N}{L^D})^{1/2}$. Furthermore, we take the attractive atomic interactions ($g_D = -|g_D|$) and a harmonic trap for potential ($V_{ext} = \frac{1}{2}m\omega_r^2 r^2$, where ω_r is radial trap frequency). Hence the energy can be written as

$$E \sim L^D \left[\frac{\hbar^2}{2m}\left(\frac{N^{1/2}}{L^{D/2+1}}\right)^2 + \frac{1}{2}m\omega_r^2\,L^2\frac{N}{L^D} - \frac{1}{2}|g_D|\left(\frac{N}{L^D}\right)^2 \right] = c_{kin}\frac{N}{L^2} + c_{pot}NL^2 - c_{int}\frac{N^2}{L^D},$$

where c_{kin}, c_{pot} and c_{int} are positive constants.

2.2.1 1D condensate

In one dimension, the expression for energy is

$$E \sim c_{kin}\frac{N}{L^2} + c_{pot}NL^2 - c_{int}\frac{N^2}{L}.$$

The kinetic energy, which behaves as $1/L^2$, prevails for small condensate sizes, whereas potential energy ($\sim L^2$) is dominant for big L, hence the energy will have a minimum at finite condensate size, as we can see in Figure 1. That localized state is a 1D matter-wave soliton.

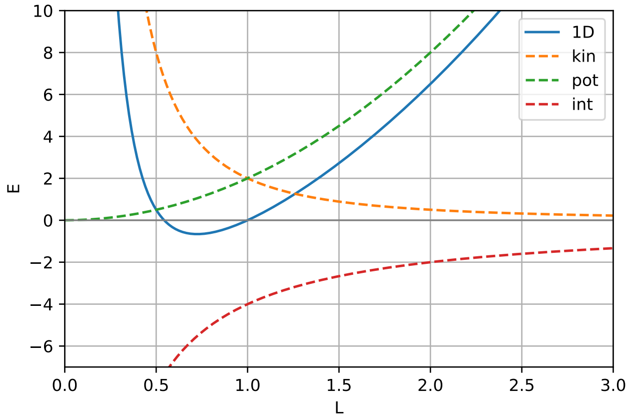


Figure 1: 1D BEC has a stable localised state at finite size.

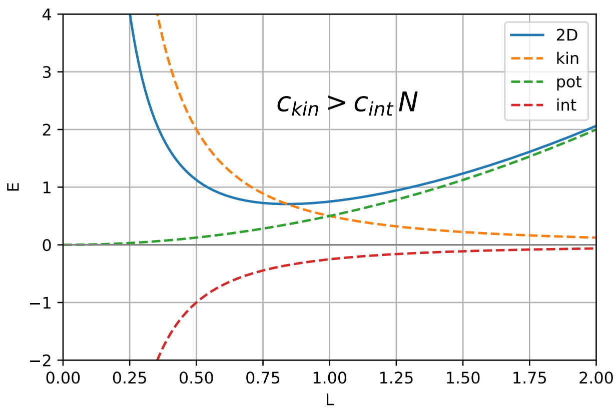
It is worth noting that the interaction term is scaled by N^2 and the others only by N. For larger number of atoms, the stable BEC size gets increasingly smaller, since the interaction term moves the minimum to the left. Intuitively, the same thing holds for bigger c_{int} due to stronger interaction.

The **kinetic energy** that stabilizes the system comes from the **ground state** (also called zero-point) energy of the quantum mechanical system and originates in Heisenberg uncertainty principle. The nonzero ground state energy is due to the **trapping potential**, which localizes the atoms and thus increases their kinetic energy. [5] Remarkably, the trap not only prevents BEC from expanding, but also from collapsing.

2.2.2 2D condensate

In two dimensions the rearranged expression for energy is

$$E \sim \frac{N(c_{kin} - N\,c_{int})}{L^2} + c_{pot}NL^2.$$



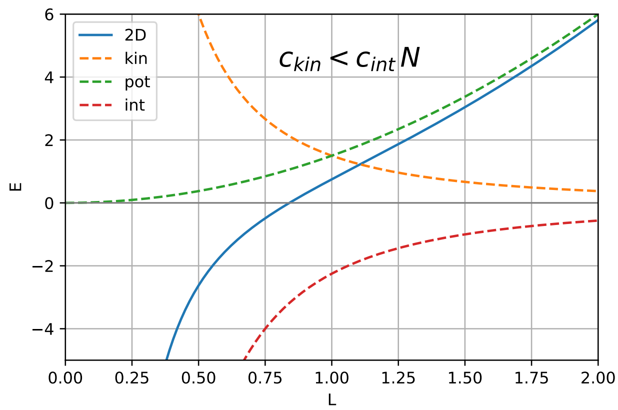


Figure 2: If the number of atoms is small enough and the interaction weak, 2D BEC has a stable size (left), otherwise the minimum appears at the origin and collapse occurs (right).

For stable solutions, the first term has to be positive, otherwise we get a global minimum at the origin (Figure 2), which means the collapse of BEC. [6] Consequently there exists a critical number of atoms N_c under which the condensate is stable, and it is inversely proportional to the strength of interaction.

2.2.3 3D condensate

Lastly, in three dimensional systems the energy is estimated as

$$E \sim c_{kin}\frac{N}{L^2} + c_{pot}NL^2 - c_{int}\frac{N^2}{L^3}.$$

Here the interaction term dominates at small sizes, so there is always a global minimum at the origin, but it is possible to generate a metastable state of finite size. For the existence of such a state a balance of all three terms is needed, which can be conveniently characterized with dimensionless **interaction parameter** $k = N|a_s|/a_r$, where $a_r = \sqrt{\hbar/m\omega_r}$ is the radial harmonic oscillator length of the trap. [6] Collapse occurs when k exceeds a critical value k_c, that is when the number of atoms is too large or the interaction is too strong, similar as in 2D case (Figure 3). A numerical value for k_c, obtained experimentally, with numerical simulations or using the Gaussian ansatz for the wavefunction, is reported to be around 0.5. [5]

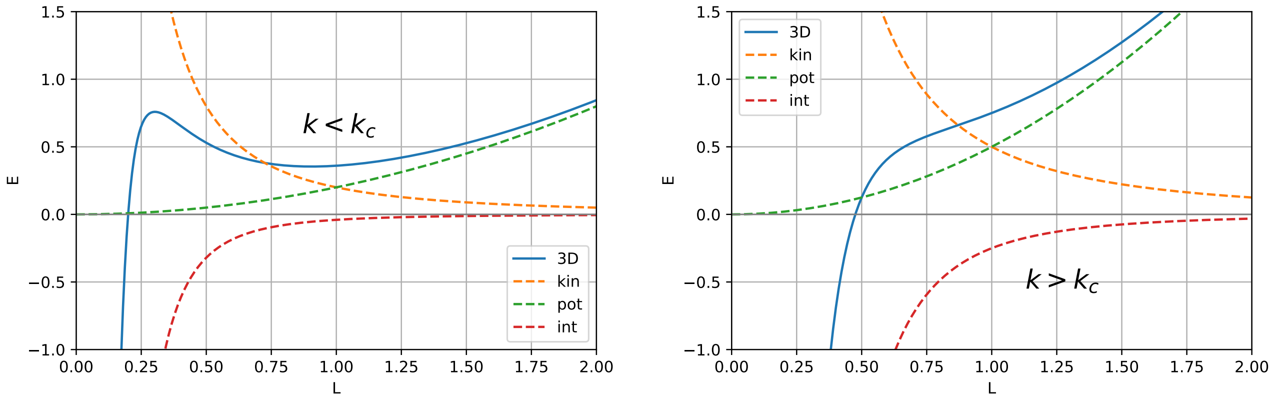


Figure 3: 3D BEC has a metastable size if the number of atoms is small enough and the interaction weak (left), otherwise the minimum appears at the origin and collapse occurs (right).

Note that the presence of external trapping potential is crucial for the existence of stable states in BEC, as mentioned earlier. Untrapped BEC with attractive interactions is always unstable to collapse. [5]

2.3 1D limit of GPE

Mathematically, **integrability** of the equation of motion corresponds to the ability of exact soliton solutions to survive mutual collisions unchanged. The Gross-Pitaevskii equation is integrable in 1D limit with $V_{ext} = 0$, which means that the amplitudes and velocities of the solitons are conserved. [7]

For the derivation of quasi-1D GPE in strong transverse confinement we assume the form of the wavefunction in cylindrical coordinates (r, φ, z) to be

$$\psi(\mathbf{r},t) = \psi(z,t)\exp\left(-\frac{r^2}{2a_r^2}\right),$$

where $a_r = \sqrt{\hbar/m\omega_r}$ is the radial harmonic oscillator length of the trap, as mentioned above. This ansatz is plugged into the original 3D GPE and the equation is integrated over r and φ to obtain the equation for $\psi(z,t)$. This wavefunction is normalized as

$$\int |\psi(z,t)|^2 \mathrm{d}z = N$$

and therefore the interaction term in the equation is changed. The external potential $V_{ext}(\mathbf{r})$ is written as

$$V_{ext}(\mathbf{r}) = \frac{1}{2} m\omega_r^2 r^2 + V_{ext}(z)$$

and its first term gives us constant shift in energy in the **1D Gross-Pitaevskii equation**:

$$i\hbar\frac{\partial}{\partial t}\psi(z,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + V_{ext}(z) + \hbar\omega_r + g_{1D}\left|\psi(z,t)\right|^2 \right]\psi(z,t),$$

where $g_{1D} = 2\, a_s\, \omega_r\, \hbar$. [8]

2.3.1 Solitonic solution and stability

If we set $V_{ext}(z) = 0$ and neglect the constant energy shift $\hbar\omega_r$, the 1D GPE has the well-known exact soliton solution

$$\psi(z,t) = \sqrt{\frac{|a_s|}{2}\frac{N}{a_r}} \exp\left[i\frac{mv}{\hbar}z - \frac{i}{\hbar}\left(\frac{mv^2}{2} - \frac{\hbar^2\kappa^2}{2m}\right)t \right] \frac{1}{\cosh\kappa\,(z - vt)},$$

where

$$\kappa = \frac{|a_s|N}{a_r^2} = \frac{k}{a_r}$$

is the inverse width of the soliton and v its velocity. [9] We get this expression for parameter κ after plugging the ansatz in the equation. The solution is correctly normalized to N.

Since we know that $k_c \approx 0.5$ and $k < k_c$ for stable condensate, we can write $\kappa^{-1} \gg a_r$. Therefore, in the 1D limit the longitudinal size of the soliton is much bigger that its radial size which corresponds to harmonic oscillator length of the radial trap.

Theoretically, 1D solutions are always stable, as we have seen in section 2.2.1. However, as the interaction parameter k increases (number of atoms grows or the interaction gets stronger) the minimum in Figure 1 moves to the left and the longitudinal size of the soliton κ^{-1} decreases. When the radial and longitudinal sizes become comparable, soliton is in 3D regime rather than 1D and is unstable to collapse. [5] So there is no guaranteed stability for solitons in quasi-1D experimental configurations - in reality only metastable states can be achieved.

3 Behaviour of solitary waves in experiments

In experiments, use of **external potentials** breaks the integrability discussed above, and only **quasi-1D** systems can be realized, so we get solitary waves (SWs) instead of true solitons, as mentioned in the introduction. The SW experiments presented in this seminar are performed in elongated BECs with a strong external confinement in two transversal dimensions. In the axial dimension there is usually a weak harmonic trap with potential $V_{ext}(z) = \pm\frac{1}{2}m\omega_z^2 z^2$, which, depending on a sign, is called trapping or anti-trapping potential.

3.1 Formation of one or multiple SWs

In this section the formation of solitary waves in regular BEC is described. The mechanism of atom cooling and trapping to create and manipulate BECs is described elsewhere [10], as well as physics behind the tuning of atomic interactions. [5] The creation of a SW is confirmed by releasing it into a weak anti-trapping potential and observing its non-dispersive propagation over macroscopic distances, as seen in Figure 4.

Firstly, BEC with repulsive interactions is created in an elongated harmonic trap with strong radial confinement (or created in an isotropic trap which is then transformed into elongated one [2]). In the next step the scattering length is tuned to a small negative value. At this point the number of atoms in the condensate exceeds the critical number, so the condensate becomes unstable to **collapse**. The actual mechanism of collapse are **three-body atomic losses** [6] which lower the number of atoms in BEC. During the collapse the interparticle distances in condensate are decreasing, which enables three atoms to come close to each other. Then two of them can form a molecule and the third atom receives the released energy as kinetic energy. In this process, all three atoms are lost, since the energy is larger that typical trap depth. Due to primary collapse, eventually the number of atoms falls under the critical number and the condensate is **stabilized**, forming a SW. As an intuitive consequence of this process, SWs generally contain about critical number of atoms and are close to 3D geometry. [5]

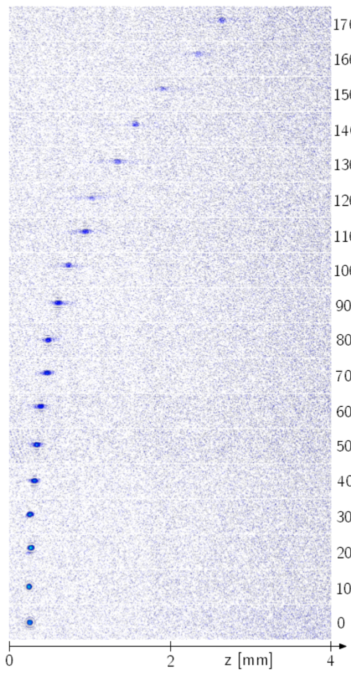


Figure 4: Propagation of SW in anti-trapping potential without dispersion over the distance of 2.9 mm in 170 ms. Courtesy of Cold atom laboratory at Jožef Stefan Institute.

Another possible outcome of BEC collapse is a so-called **soliton train**, containing multiple SWs, which was first observed in 2002. [11] In this case, the number of atoms remaining in the condensate is higher than the critical number, but they are divided into multiple distinct SWs with atom number just under N_c (example in Figure 5). Experiments show that these SWs are remarkably **stable**, persisting for many cycles of oscillation in a harmonic trap despite being near the threshold for collapse. [11] This stability is a consequence of **relative phase** π between SWs. As explained later in the seminar, the dynamics of interactions between SWs are determined by their relative phase $\Delta\phi$. For $\Delta\phi = 0$, coherent overlap of solitons can occur, resulting in secondary collapse if the number of atoms temporarily exceeds the critical one. [12] Soliton trains being stable therefore implies interaction between SWs with relative phase π, which ensures that conditions for secondary collapse are never met.

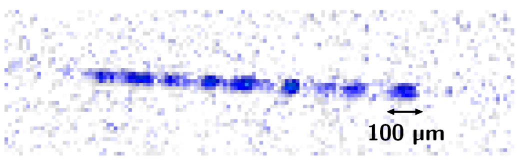


Figure 5: Soliton train. Courtesy of Cold atom laboratory at Jožef Stefan Institute.

The mechanism responsible for the soliton train formation is **modulational instability** (MI). As the scattering length in BEC is rapidly changed from positive to negative, MI causes the exponential growth of small density fluctuations into density modulations in the condensate. Atoms move into spots with increased density and evolve into solitons. [13] There were two theories regarding how π-phase differences are formatted, ensuring the stability of soliton trains. Firstly, it was proposed that quantum fluctuations seed the MI and during collapse imprint the condensate with phase structure, restricting phase difference to values close to π. [5] Another idea was that the perturbations for MI originate in self-interference of the condensate. In this case it was found that SWs are created with arbitrary phases and only after series of secondary collapses, induced by collisions of in-phase ($\Delta\phi = 0$) SWs, they settle into stable out-of-phase configuration. [12]

In 2017, it was concluded [13] that modulational instability is driven by **noise**, but it is not yet known whether it is quantum or not. For small $|a_s|$ (corresponding to larger number of atoms in individual SWs), it was surprisingly discovered that neither primary nor secondary collapses have occurred during the soliton train formation. Such SWs were already out-of-phase during the formation of the soliton train. For larger $|a_s|$ though, both primary and secondary collapses were present, which is why the initial relative phase could not be observed.

3.2 Collisions of two SWs

The defining property of true solitons is, apart from being non-dispersive, their ability to **pass through one another** with unchanged velocity, amplitude or shape, but possibly with altered trajectory due to a phase shift. [14] Solitary waves however, although created in quasi-1D geometries, are often created close to the transition between 1D and 3D due to the nature of experimental process, described in the previous section. As a consequence, some non-solitonic behaviour is manifested in SW collisions. They are a complex phenomenon with properties depending heavily on the interaction parameter k, velocity of SWs and relative phase between them.

As shown in one of several theoretical analyses [15] and sole experimental work [14], relative phase $\Delta\phi$ is crucial for the collisional dynamics. For $\Delta\phi = 0$ or **in-phase** collision, wavepackets overlap and form a **density peak**, which does not appear for out-of-phase $\Delta\phi = \pi$, as can be seen on Figure 6.

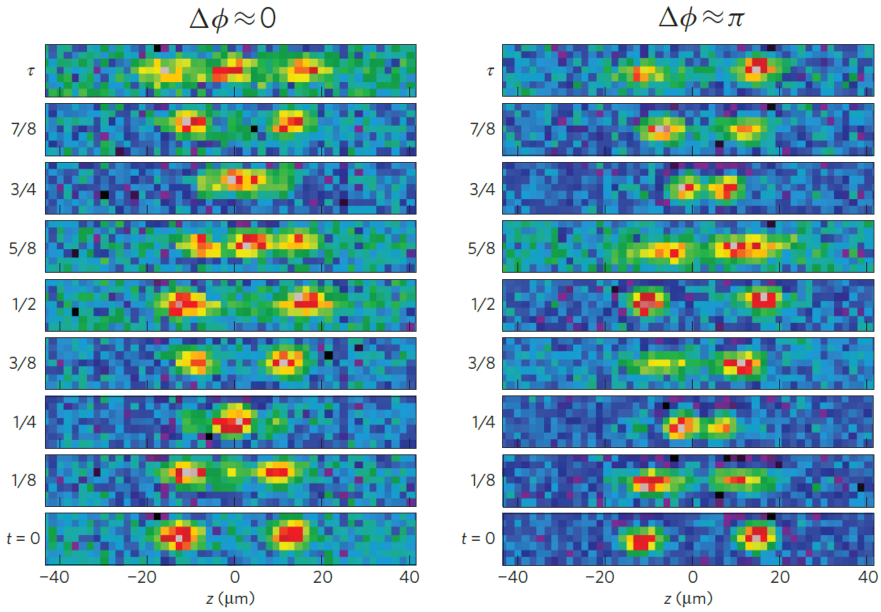


Figure 6: Phase-dependent collisions of SWs in harmonic trap. An in-phase collision has a density peak at the centre of mass (left), an out-of-phase collision does not (right). Adapted from [14].

In the density peak, distinctive for $\Delta\phi = 0$, the number of atoms can increase above the threshold for collapse, causing collapse instability. That causes the apparent annihilation of in-phase SW pair or reduction of the atom number in SWs, if only partial collapses occur. On the other hand, SW pairs with $\Delta\phi = \pi$ are remarkably stable and survive many oscillations in a trap. [14] That is why relative phase π between SWs is believed to be the reason for stability of soliton trains, as stated earlier. Those collisions are generally stable even though the number of atoms in both SWs together exceeds the critical number.

The result of SWs overlap is wave **interference pattern**, which can be nicely seen in Figure 7, the result of numerical simulations of 3D GPE. Figures (b)(ii) and (c)(ii) seem to match well with Figure 6, since density peak or lack thereof is clearly recognizable. Additionally, the effect of SWs velocity can be observed. The number of collisional interference fringes increases with velocity and is, though hard to see on upper figures, always odd for $\Delta\phi = 0$ and even for $\Delta\phi = \pi$. Lower figures also follow this rule with 1 or 0 fringe, respectively. On figure 7(a) it is also shown how stability of SWs with different relative phases depends on interaction parameter k - for small enough velocities, collisions with $\Delta\phi = \pi$ are unsurprisingly much more stable.

However, as the velocity increases, stability depends less and less on $\Delta\phi$ and approaches critical value k_c for an isolated SW. To explain this, it was proposed that there is a **characteristic time** t_{col} for collapse to occur [15]. If the time t_{int} of two SWs overlapping is much shorter that t_{col}, there is not

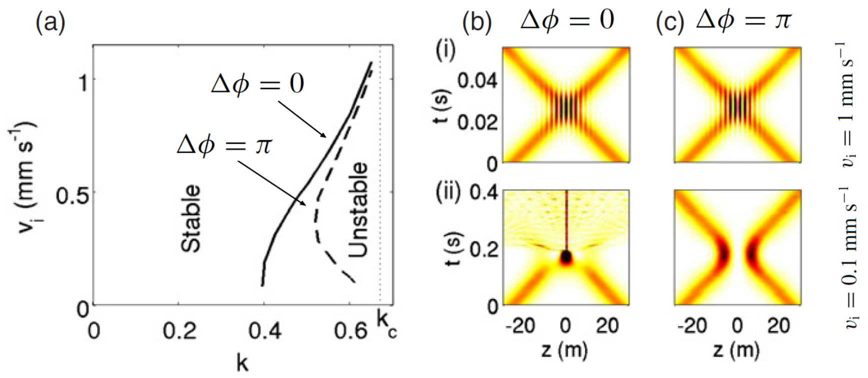


Figure 7: (a) Phase space for stability of SWs, colliding in a waveguide without axial potential. Stability depends on interaction parameter k and velocity of SWs v_i. (b)-(c) Evolution of SW collisions for different parameters with clearly visible interference fringes. Adapted from [15].

enough time for the collapse to happen and SWs pass one another unchanged. Interaction time t_{int} is inversely proportional to velocity and that is why for large velocities stability is easily achieved. There is clearly no such limitation for the interference, so it appears for larger velocities as well.

A natural question regarding Figure 7(c)(ii) is whether that is just an interference pattern looking like a reflection or do SWs actually repel. The interpretation in [11] and [16] is that $\Delta\phi = \pi$ prevents overlapping with effective repulsive force such that SWs rebound rather than pass through each other. It is known that in the 1D limit, the force between two solitons depending on $\Delta\phi$ changes continuously from attractive to repulsive. [15] This transition can be seen in Figure 8(a), which does the same as 7(b) and 7(c), but for 1D NLS equation.

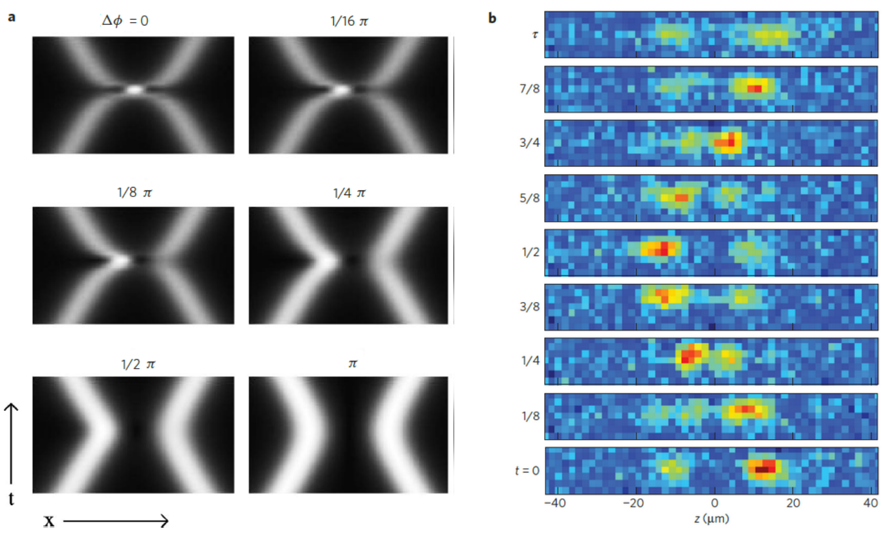


Figure 8: (a) Numerical simulation of 1D NLS resulting in phase dependent soliton interactions. (b) Collision of SWs with 2:1 ratio of atom number. Adapted from [17] and [14].

However, in Figure 8(b) there is a convincing experiment that offers proof of SWs passing through one another with $\Delta\phi = \pi$: trajectories of two different SWs. Relative phase is π, since no density peak appears between the SWs during the collision. The authors argue that interference pattern of two passing solitons with $\Delta\phi = \pi$ only gives the **appearance of reflection**. The possibility of SWs exchanging particles during collisions was found in [15], but ruled out as an explanation since that happens at much lower velocities.

Finally, let's go back to **altered trajectories** mentioned in the beginning of this section. They can be seen in Figure 8(a) - the incident trajectories are not aligned with the ones that go out, which is one of the general differences between nonlinear and linear interactions. Interestingly, this effect was measured in [14] - they observed SWs oscillating with frequency higher than the trap frequency ω_z, which is the consequence of such trajectory jump in a trap. They found the frequency shift **independent of $\Delta\phi$** and gave an intuitive explanation: since interaction among atoms is attractive, atoms accelerate the SWs as they get close one another and decelerate them back to original velocity while they are moving away. Hence SWs need less time to complete the movement in the trap and have higher frequency.

4 Conclusion

Emphasis of this seminar was on the stable configurations of Bose-Einstein condensates. The discussion started with general stability of BECs in a trap, continued with stability of single solitary matter-wave, then multiple SWs in a soliton train and lastly stability of two SWs during multiple collisions. Being able to create states that are stable and also free from dispersion is a huge advantage, since those properties are promising for a wide range of applications, such as atom interferometry, atom sensors for high-precision measurements and quantum-information processing. [5] It was proposed that pulsed atomic soliton laser could be made with simple adaptation of existing setups, in which all collapses could be avoided. [18] To conclude, solitary matter-waves in Bose-Einstein condensates are a fascinating topic with a lot of open experimental challenges, potentially leading to important advances in quantum technology.

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